On an Inverse Eigenvalue Problem for Unitary Hessenberg Matrices

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ABSTRACT

We show that a unitary upper Hessenberg matrix with positive subdiagonal elements is uniquely determined by its eigenvalues and the eigenvalues of a modified principal submatrix. This provides an analog of a well-known result for Jacobi matrices.

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1. INTRODUCTION

We refer to a real symmetric tridiagonal matrix with positive subdiagonal entries as a *Jacobi matrix.* Jacobi matrices are closely connected with polynomials orthogonal with respect to a measure on an interval of the real axis, and inverse eigenvalue problems for Jacobi matrices have been considered by several authors (see, for example, [8, 2] and the references therein).

Similarly, unitary upper Hessenberg matrices with positive subdiagonal elements are related to polynomials orthogonal with respect to a measure on the unit circle in the complex plane. One inverse eigenvalue problem for unitary Hessenberg matrices is considered in [1] (see Theorem 3 below), and methods for constructing a unitary Hessenberg matrix from spectral data are described in [1, 10]. In this paper we prove another inverse spectral

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property of unitary Hessenberg matrices, which is stated in Theorem 5 below.

Let $\mathcal{H}_n$ denote the set of unitary upper Hessenberg matrices of order n with positive subdiagonal elements. Any $H \in \mathcal{H}_n$ can be uniquely expressed as the product

$$H = G_1(\gamma_1)\ldots G_{n-1}(\gamma_{n-1})\tilde{G}_n(\gamma_n), \tag{1}$$

where each $\gamma_k \in \mathbb{C}$ with $|\gamma_k| < 1$ for $1 \leq k < n$, and $|\gamma_n| = 1$. Here $G_k(\gamma_k)$ denotes the Givens transformation

$$G_k(\gamma_k) = \begin{bmatrix} I_{k-1} & & & \\ & -\gamma_k & \sigma_k & \\ & \sigma_k & \overline{\gamma}_k & \\ & & & I_{n-k-1} \end{bmatrix},$$

where $\sigma_k = \sqrt{1-|\gamma_k|^2} > 0$, $k = 1, \ldots, n-1$, and

$$\tilde{G}_n(\gamma_n) = \mathrm{diag}[I_{n-1}, -\gamma_n].$$

Each $H \in \mathcal{H}_n$ is therefore determined by the $2n-1$ real parameters that compose the *Schur parameters* $\{\gamma_j\}_{j=1}^n$ of H. We refer to (1) as the *Schur parametric form* of H, and write $H = H(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$. The Schur parameterization of $\mathcal{H}_n$ is presented in [7]. It plays a fundamental role in the development of efficient algorithms for solving eigenproblems for unitary Hessenberg matrices. Observe that the complementary parameters $\{\sigma_k\}_{k=1}^{n-1}$ are the subdiagonal elements of H; i.e., $\sigma_k = e_{k+1}^T H e_k$. Throughout this paper e_j denotes the jth column of the identity matrix of appropriate size.

Let $\lambda(T)$ denote the spectrum of a square matrix T. The following two results are fundamental in the development of inverse eigenvalue problems for Jacobi matrices (see, for example, [8, 2]).

Theorem 1. *Given two strictly interlaced sets of real numbers $\{\lambda_k\}_{k=1}^n$ and $\{\mu_k\}_{k=1}^n$, there is a unique Jacobi matrix T_n and a unique real number β_0 such that $\lambda(T_n) = \{\lambda_k\}$ and $\lambda(\tilde{T}_n) = \{\mu_k\}$, where $\tilde{T}_n := T_n + \beta_0 e_1 e_1^T$.*

Theorem 2. *Given $2n-1$ real numbers satisfying*

$$\lambda_1 < \mu_1 < \lambda_2 < \ldots < \mu_{n-1} < \lambda_n,$$

there exists a unique Jacobi matrix T_n such that $\lambda(T_n) = \{\lambda_k\}$ and $\lambda(T_{n-1}) = \{\mu_k\}$, where T_{n-1} denotes the leading principal submatrix of T_n of order $n-1$.

Theorem 1 states that a Jacobi matrix T_n is uniquely determined by its eigenvalues and the eigenvalues of an additive rank-one perturbation $\tilde{T}_n$, which differs from T_n only in the (1,1) entry. The following unitary analog of Theorem 1 is proved in [1]. It states that $H \in \mathcal{H}_n$ is uniquely determined by its eigenvalues and the eigenvalues of a multiplicative rank-one perturbation of H. This perturbation differs from H by a multiplicative factor α in the first row only.

THEOREM 3. *Given two sets of strictly interlaced points $\{\lambda_k\}_{k=1}^n$ and $\{\mu_k\}_{k=1}^n$ on the unit circle, there exists a unique $H = H(\gamma_1, \ldots, \gamma_n) \in \mathcal{H}_n$ and a unique $\alpha \in \mathbb{C}$ of unit modulus such that $\lambda(H) = \{\lambda_k\}_{k=1}^n$ and $\lambda(\tilde{H}) = \{\mu_k\}_{k=1}^n$, where $\tilde{H} = H(\alpha\gamma_1, \ldots, \alpha\gamma_n) = (I - (1-\alpha)e_1 e_1^T)H$.*

Of course, the leading principal submatrix H_{n-1} of H is not unitary, and its eigenvalues do not lie on the unit circle. An obstacle to the formulation of an inverse eigenproblem for unitary Hessenberg matrices that is analogous with Theorem 2 is the identification of an appropriate modification of H_{n-1}. In particular, the modified principal submatrix H'_{n-1} should be unitary, and its spectrum should interlace that of H on the unit circle. Recent work by Delsarte and Genin [4] [5] provides such modifications of H_{n-1}. These modified submatrices are used in [3] to derive a method for the unitary Hessenberg eigenproblem that is analogous with the Sturm sequence method. We use them here to formulate and prove a result for unitary matrices that is similar to Theorem 2.

2. MODIFIED SUBMATRICES

The $k \times k$ leading principal submatrix H_k of $H = H(\gamma_1, \ldots, \gamma_n) \in \mathcal{H}_n$ is given by $H_k = H(\gamma_1, \ldots, \gamma_k)$. The characteristic polynomials $\chi_k(\lambda) = \det(\lambda I_k - H_k)$ are the well-known *Szegő polynomials* associated with H. These polynomials satisfy the *Szegő recurrence relation*

$$\begin{aligned} \chi_k(\lambda) &= \lambda\chi_{k-1}(\lambda) + \gamma_k\tilde{\chi}_{k-1}(\lambda), \\ \tilde{\chi}_k(\lambda) &= \tilde{\chi}_{k-1}(\lambda) + \lambda\overline{\gamma}_k\chi_{k-1}(\lambda), \end{aligned} \qquad k = 1, \ldots, n, \tag{2}$$

where $\chi_0(\lambda) = 1$ and $\tilde{\chi}_0(\lambda) = 1$. The $\tilde{\chi}_k(\lambda)$ enter as auxiliary polynomials. However, from the recursions one can show that $\tilde{\chi}_k(\lambda) = \lambda^k\overline{\chi}_k(1/\lambda)$. See, for example, [7].

Given a sequence of numbers ρ_k with $|\rho_k| = 1$, $k = 1, \ldots, n$, we define the *modified* leading principal submatrices of H by

$$H'_k = H(\gamma_1, \ldots, \gamma_{k-1}, \rho_k).$$

Let ψ_k denote the monic characteristic polynomial of H'_k. Then it is easily seen that the ψ_k are related to the Szegő polynomials by

$$\psi_k(\lambda) = \lambda\chi_{k-1}(\lambda) + \rho_k\tilde{\chi}_{k-1}(\lambda), \quad k = 1, \ldots, n. \tag{3}$$

Observe that the roots of ψ_k are distinct and of unit modulus, since each H'_k is an unreduced unitary Hessenberg matrix.

In [4, 5], it is shown that the parameters ρ_k can be chosen so that the polynomials $\{\psi_k\}_{k=0}^n$ satisfy a three-term recurrence relation, and so that the roots of ψ_k interlace those of ψ_{k+1} on the unit circle. These properties are used in [3] to develop an analog of the Sturm sequence method for finding the eigenvalues of a unitary Hessenberg matrix. The particular choice of the parameters ρ_k is given in the following theorem.

THEOREM 4 ([3]) *Let $H = H(\gamma_1, \ldots, \gamma_n) \in \mathcal{H}_n$, and assume that $-1 \notin \lambda(H)$. Define*

$$\begin{aligned} \rho_n &= \gamma_n \\ \rho_k &= \frac{\gamma_k - \rho_{k+1}}{1 - \overline{\gamma}_k\rho_{k+1}}, \quad k = n-1, \ldots, 1. \end{aligned} \tag{4}$$

Let $\{\lambda_j^{(k)}\}_{j=1}^k$ denote the eigenvalues of H'_k, ordered with increasing argument in $(-\pi, \pi]$; i.e. $\lambda_j^{(k)} = \exp(i\theta_j^{(k)})$, where $-\pi < \theta_1^{(k)} < \theta_2^{(k)} < \ldots < \theta_k^{(k)} \leq \pi$. Then for each $k = 1, \ldots, n-1$,

$$-\pi < \theta_1^{(k+1)} < \theta_1^{(k)} < \theta_2^{(k+1)} < \ldots < \theta_k^{(k)} < \theta_{k+1}^{(k+1)} < \pi.$$

In other words, if we order the eigenvalues counterclockwise on the unit circle with cutting point -1, then the eigenvalues of H'_k strictly interlace those of H'_{k+1} for each $k = 1, \ldots, n-1$.

We state the above result using -1 as the cutting point of the unit circle for notational convenience. This parameter, which is called the *circle parameter* in [4, 5], can be chosen to be any unimodular number $\mu_0 \notin \lambda(H)$. In this more general case, Theorem 4 remains valid with $\rho_k = \dfrac{\gamma_k + \overline{\mu}_0\rho_{k+1}}{1 + \overline{\mu}_0\overline{\gamma}_k\rho_{k+1}}$. This interlacing result allows us to formulate our main result, which is a unitary analog of Theorem 2.

THEOREM 5. *Let $\{\lambda_k\}_{k=1}^n$ and $\{\mu_k\}_{k=0}^{n-1}$ be two sets of strictly interlacing points on the unit circle in the complex plane with respect to a cutting point μ_0. Then there exists a unique $H = H(\gamma_1, \ldots, \gamma_n) \in \mathcal{H}_n$ such that $\lambda(H) = \{\lambda_k\}_{k=1}^n$ and $\lambda(H'_{n-1}) = \{\mu_k\}_{k=1}^{n-1}$, where $H'_{n-1} = H(\gamma_1, \ldots, \gamma_{n-2}, \rho_{n-1})$ and $\rho_{n-1} = \dfrac{\gamma_{n-1} + \overline{\mu}_0\gamma_n}{1 + \overline{\mu}_0\overline{\gamma}_{n-1}\gamma_n}$.*

3. UNIQUENESS OF THE REDUCTION

The following well-known result is fundamental for our approach to the inverse eigenproblem, and follows from the result referred to as the implicit Q Theorem in [6, §7.4.5].

LEMMA 6. *Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be a unitary diagonal matrix with distinct λ_k, and let $w_1 = [\omega_k]_{k=1}^n \in \mathbb{C}^n$ be a unit vector with each $\omega_k \neq 0$. Then there exists a unique unitary matrix W and a unique $H \in \mathcal{H}_n$ such that $We_1 = w_1$ and $\Lambda W = WH$.*

Observe that if H is determined by (Λ, w_1), then the pair (Λ, Dw_1) also determines H for any unitary diagonal matrix D (i.e., the normalized eigenvectors of H are unique up to a unimodular scaling). We can therefore require that each ω_k be positive. Moreover, we see that any unitary Hessenberg matrix with positive subdiagonal elements is uniquely determined by its eigenvalues and the moduli of the first components of its normalized eigenvectors.

The Stieltjes procedure for Szegő polynomials presented in [10] provides an efficient implementation of the isometric Arnoldi process for constructing H from Λ and w_1. Another procedure for the construction of H from its eigenvalues Λ and associated weights w_1 is the inverse unitary QR (IUQR) algorithm [1]. The latter method is based on unitary similarity transformations and provides an efficient algorithm that is more robust than the Stieltjes procedure for Szegő polynomials. The IUQR algorithm is implemented in terms of the Schur parametrization of $\mathcal{H}_n$, and is similar in form to the algorithm for the construction of Jacobi matrices presented in [8].

It is easily seen that $H \in \mathcal{H}_n$ is also uniquely determined by its eigenvalues and the *last* components of its normalized eigenvectors. This follows by applying Lemma 6 to determine the matrix $H^P = JH^TJ \in \mathcal{H}_n$, where J is the reversal permutation matrix, $J = [e_n, e_{n-1}, \ldots, e_1]$. The Schur parameters of $H = H(\gamma_1, \ldots, \gamma_n)$ and $H^P = H(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)$ are related by

$$\tilde{\gamma}_k = \overline{\gamma}_{n-k}\gamma_n, \quad k = 1, \ldots, n-1, \quad \gamma_n = \tilde{\gamma}_n, \tag{5}$$

and the spectral decomposition of H^P is

$$\Lambda Q = QH^P,$$

where $Q = \overline{W}J$.

4. SOLUTION OF THE INVERSE EIGENVALUE PROBLEM

We prove Theorem 5 through the following sequence of lemmas, in which we derive formulas for the last components of the normalized eigenvectors of H in terms of $\psi_{n-1}(\lambda)$ and $\chi_n'(\lambda)$. Our use of the last components stems from our use of the characteristic polynomial $\psi_{n-1}(\lambda)$ of the modified leading principal submatrix as described in [3]. One can easily re-formulate these results using the first components of the eigenvectors and the characteristic polynomial of a modified trailing principal submatrix.

LEMMA 7. *Suppose that $H \in \mathcal{H}_n$ has the spectral decomposition $H = Q^*\Lambda Q$. Then the squared moduli of the components of q_n, the last column of Q, are given by*

$$|q_{jn}|^2 = \frac{\chi_{n-1}(\lambda_j)}{\chi_n'(\lambda_j)}, \quad j = 1, \ldots, n$$

where $\chi_{n-1}(\lambda)$ and $\chi_n(\lambda)$ are the monic Szegő polynomials of $H(\gamma_1, \ldots, \gamma_n)$ of degree $n-1$ and n, respectively, and χ_n' is the derivative of χ_n.

Proof. Observe that

$$e_n^T(\lambda I - H)^{-1}e_n = \frac{\det(\lambda I - H_{n-1})}{\det(\lambda I - H)} = \frac{\chi_{n-1}(\lambda)}{\chi_n(\lambda)}.$$

Using the spectral decomposition, we have

$$\frac{\chi_{n-1}(\lambda)}{\chi_n(\lambda)} = q_n^*(\lambda I - \Lambda)^{-1}q_n = \sum_{j=1}^{n} \frac{|q_{jn}|^2}{\lambda - \lambda_j}.$$

Since $\chi_n(\lambda_j) = 0$ for each j, we have

$$|q_{jn}|^2 = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j)\frac{\chi_{n-1}(\lambda)}{\chi_n(\lambda) - \chi_n(\lambda_j)} = \frac{\chi_{n-1}(\lambda_j)}{\chi_n'(\lambda_j)}. \blacksquare$$

LEMMA 8. *Let $\psi_{n-1}(\lambda)$ denote the characteristic polynomial of $H_{n-1}' = H(\gamma_1, \ldots, \gamma_{n-2}, \rho_{n-1})$, where $\rho_{n-1} = \dfrac{\gamma_{n-1} + \overline{\mu}_0\gamma_n}{1 + \overline{\mu}_0\overline{\gamma}_{n-1}\gamma_n}$. Then*

$$\chi_n(\lambda) = \frac{\gamma_n\sigma_{n-1}^2}{\rho_{n-1} - \gamma_{n-1}}\psi_{n-1}(\lambda) + (\lambda - \mu_0)\chi_{n-1}(\lambda).$$

Proof. From the recurrence relations (2) and (3) with $k = n-1$, we have

$$\chi_{n-1}(\lambda) = \lambda\chi_{n-2}(\lambda) + \gamma_{n-1}\tilde{\chi}_{n-2}(\lambda),$$

$$\psi_{n-1}(\lambda) = \lambda\chi_{n-2}(\lambda) + \rho_{n-1}\tilde{\chi}_{n-2}(\lambda),$$

so that

$$\tilde{\chi}_{n-2}(\lambda) = \frac{\psi_{n-1}(\lambda) - \chi_{n-1}(\lambda)}{\rho_{n-1} - \gamma_{n-1}},$$

$$\lambda\chi_{n-2}(\lambda) = \frac{\rho_{n-1}\chi_{n-1}(\lambda) - \gamma_{n-1}\psi_{n-1}(\lambda)}{\rho_{n-1} - \gamma_{n-1}}.$$

Substituting the above two expressions into

$$\tilde{\chi}_{n-1}(\lambda) = \tilde{\chi}_{n-2}(\lambda) + \lambda\overline{\gamma}_{n-1}\chi_{n-2}(\lambda),$$

we obtain

$$\begin{aligned}
\tilde{\chi}_{n-1}(\lambda) &= \frac{\sigma_{n-1}^2}{\rho_{n-1} - \gamma_{n-1}}\psi_{n-1}(\lambda) + \frac{\overline{\gamma}_{n-1}\rho_{n-1} - 1}{\rho_{n-1} - \gamma_{n-1}}\chi_{n-1}(\lambda) \\
&= \frac{\sigma_{n-1}^2}{\rho_{n-1} - \gamma_{n-1}}\psi_{n-1}(\lambda) - \mu_0\overline{\gamma}_n\chi_{n-1}(\lambda).
\end{aligned}$$

The result then follows from the above expression and $\chi_n(\lambda) = \lambda\chi_{n-1}(\lambda) + \gamma_n\tilde{\chi}_{n-1}(\lambda)$. ■

Since $\chi_n(\lambda_j) = 0$, Lemma 8 yields

$$\chi_{n-1}(\lambda_j) = \frac{-\gamma_n\sigma_{n-1}^2}{\rho_{n-1} - \gamma_{n-1}}\;\frac{\psi_{n-1}(\lambda_j)}{(\lambda_j - \mu_0)}.$$

This immediately implies the following lemma.

LEMMA 9. *The squared moduli of the last components of the eigenvectors of H are given by*

$$|q_{jn}|^2 = \alpha\frac{\psi_{n-1}(\lambda_j)}{(\lambda_j - \mu_0)\chi_n'(\lambda_j)} \quad j = 1, \ldots, n, \tag{6}$$

where $\alpha = \dfrac{-\gamma_n\sigma_{n-1}^2}{\rho_{n-1} - \gamma_{n-1}}$.

We are now in position to prove Theorem 5. Note that the complex number α serves to make (6) nonnegative for each j, and also to normalize the weights so that $\sum_{j=1}^{n} |q_{jn}|^2 = 1$. Consequently, we see that

$$|q_{jn}| = |\xi_j|/\|x\|_2, \tag{7}$$

where $x = [\xi_1, \ldots, \xi_n]^T$ and

$$\xi_j^2 = \frac{\psi_{n-1}(\lambda_j)}{(\lambda_j - \mu_0)\chi_n'(\lambda_j)} = \frac{\prod_{k=1}^{n-1}(\lambda_j - \mu_k)}{(\lambda_j - \mu_0)\prod_{\substack{k=1\\k\neq j}}^{n}(\lambda_j - \lambda_k)}, \tag{8}$$

for $j = 1, \ldots, n$. It follows that

$$|\xi_j| = \left| \frac{\prod_{k=1}^{n-1} \sin\big((\nu_k - \theta_k)/2\big)}{\sin\big((\nu_0 - \theta_j)/2\big) \prod_{\substack{k=1\\k\neq j}}^{n} \sin\big((\theta_k - \theta_j)/2\big)} \right|^{1/2}, \tag{9}$$

where $\lambda_k =: \exp(i\theta_k)$, $k = 1, \ldots, n$, and $\mu_k =: \exp(i\nu_k)$, $k = 0, \ldots, n-1$. These formulas show how the last components of the normalized eigenvectors of $H \in \mathcal{H}_n$ can be constructed from the strictly interlacing sets on the unit circle $\{\lambda_k\}_{k=1}^n$ and $\{\mu_k\}_{k=0}^n$. Theorem 5 then follows from the discussion following Lemma 6.

In summary, the following procedure can be used to construct the solution $H = H(\gamma_1, \ldots, \gamma_n)$ of the inverse eigenvalue problem posed in Theorem 5.

(1) Compute $|\xi_j|$ by (9) for $j = 1, \ldots, n$, and obtain $w_1 = [\,|q_{jn}|\,]_{j=1}^n$ by (7).
(2) Use the IUQR Algorithm of [1] to compute the Schur parameters $\{\tilde{\gamma}_k\}$ of the unique $\tilde{H} \in \mathcal{H}_n$ such that $\Lambda W = W\tilde{H}$, where $\tilde{H} = G_1(\tilde{\gamma}_1) \ldots G_n(\tilde{\gamma}_n)$, and $W = (w_1, w_2, \ldots, w_n)$ is unitary.
(3) Determine the Schur parameters of $H := \tilde{H}^P$ from (5).

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